DYNKIN OPERATORS, RENORMALIZATION AND THE GEOMETRIC β FUNCTION

SUSAMA AGARWALA

Abstract. In this paper, I show a close connection between renormalization and a generalization of the Dynkin operator in terms of logarithmic derivations. The geometric β function, which describes the dependence of a Quantum Field Theory on an energy scale defines is defined by a complete vector field on a Lie group G defined by a QFT. It also defines a generalized Dynkin operator.

1. Introduction

The Dynkin operator has recently become an important object in the study of dynamical systems. The classical Dynkin operator defines a bijection from a Lie group to it's Lie algebra, the inverse of the exponential map. It is key in the closed form exapansion of the Baker-Campbell-Hausdorff formula. In [11], the authors generalize Dynkin operators in terms of logarithmic derivatives on a Lie algebra, and connect it to Magnus-type formulas. The classical Magnus formula provides a solution to the system of differential equations of the form

$$X'(t) = A(t)X(t) \; . \tag{1}$$

Systems of this form appear in the study of renormalization of quantum filed theories (QFTs). In [5], the authors define a β function, a Lie algebra element representing how a dimensionally regularized QFT depends on the energy scale. The β function for dimensional regularization and momentum cutoff regularization satisfies an equation of the form (1). In [6, 2], the authors show that this β function defines a connection that also satisfies (1). In this note, I show that there is a much deeper connection between the Dynkin operator and renormalization.

As in the literature on the Hopf algebraic approach to renormalization, intiated by [4], consider a regularized perturbative Quantum Field Theory (QFT), ϕ, as a map from Feynman diagrams to an algebra $\mathcal{A}$. The divergence structure of the Feynman diagrams is encoded in a Hopf algebra $\mathcal{H}$, as initially introduced by Connes and Kreimer in [4]. I wish to keep the discussion in the paper general, but for specific examples, one can consider the Hopf algebra structure on scalar field theories, developed in [4], on QED developed in [12], on gauge theories developed in [10]. The algebras in all these cases have been the algebra of formal Laurent series, $\mathcal{A} = \mathbb{C}[z^{-1}][[z]]$ [3]. However, if one is interested in momentum cut-off renormalization, $\mathcal{A} = \mathbb{C}[\log z, z^{-1}][[z]]$ is appropriate [3]. For a scalar field theory over a curved, compact Euclidean background, use $\mathcal{A} = \mathcal{D}'(M)[z^{-1}, z]][1]$.

In section 2, I generalize the β function defined in [5, 1, 3]. I generalize regularized Feynman rules as elements of an affine Lie group associated to a Hopf algebra, $\mathcal{H}$. The action of the renormalization scale generalizes to a flow on this group. Specifically, it defines a one parameter family of diffeomorphisms. The β function defining the action of the renormalization scale action is the vector field of the flow pulled back to the Lie algebra. In section 3, I recall the Dynkin operator, $D : T(V) \to V$, a map from a tensor algebra to the underlying Lie algebra that defines a map from the Lie group $G = exp(V)$ to V. In [9], the authors showed that the β function of [5] can be written as a variation of this map on G. I generalize this map for the class of geometric β function defined in section 2. In [11], the authors define a generalization of the classical Dynkin operator using logarithmic derivatives with regards to a Lie derivative. I show that the geometric β function, as defined in [?], is compatible with the Dynkin variant defined in [9].

2. The perturbative β function

The literature on renormalization theory is often confusing because of different nomenclature referring to slightly different things in different parts of the community. To avoid this confusion, I use this section to

set up a dictionary of what I mean when I use different terms commonly found in the physics literature, and what mathematical generalizations they correspond to. In this way, I motivate why the definition of a geometric β function is the appropriate object of study.

Definition 1. The Hopf algebra of Feynman diagrams, $\mathcal{H}$ is a commutative Hopf algebra over a field k of characteristic 0 associated to the Feynman diagrams for some QFT, as originally constructed in [4]. The Hopf algebra is constructed to encode the subdivergence structure of the Feynman integrals in a manner that is compatible with BPHZ renormalization.

Recall a few useful properties of a Hopf algebra of Feynman diagrams. The Hopf algebra $\mathcal{H}$ is generated by all 1PI graphs of the QFT. For more details on this Hopf algebra, see [4, 7, 2], The coproduct of a graph $\Gamma \in \mathcal{H}$ is given by

$$\Delta(\Gamma) = \sum_{\substack{\gamma \subseteq \Gamma \\ \gamma, \Gamma/\gamma \in \mathcal{H}}} \gamma \otimes \Gamma/\gamma \; ,$$

where Γ/γ is the obtained from Γ by the contraction of the connected components of γ to a point. This coproduct encodes the divergence structure found in BPHZ renormalization. Multiplication of graphs is given by disjoint union. The counit is written

$$\varepsilon(h) = \begin{cases} h & \text{h } \in \mathcal{H}_0 \\ 0 & \text{else.} \end{cases}$$

The Hopf algebra is graded by loop number, with the grading operator $Y(\Gamma) = n\Gamma$ if Γ has n loops. The antipode is defined recursively as

$$S(\Gamma) = -\Gamma - \sum_{\substack{\gamma \subseteq \Gamma \\ \gamma, \Gamma/\gamma \in \mathcal{H}}} S(\gamma)\Gamma/\gamma \; .$$

The coproduct structure on $\mathcal{H}$ induces a convolution product on the associated affine group scheme $G = \text{Spec } \mathcal{H}$. For a given k-algebra $\mathcal{A}$, the Lie group $G(\mathcal{A}) = \text{Hom}_{k\ alg}(\mathcal{H}, \mathcal{A})$. That is, for $g, g' \in G(\mathcal{A})$ and $\Gamma \in \mathcal{H}$,

$$g \star g'(\Gamma) = (g \otimes g')(\Delta\Gamma) \; .$$

Note that $\mathcal{H} \simeq k[G]$, the ring of regular functions on G.

Definition 2. In this paper, the renormalization group is G. It is the group of evaluations of the Hopf algebra of the QFT $\mathcal{H}$.

Given a QFT, there are well established Feynman rules that assign a divergent integral to each Feynman diagram. Given a regularization scheme, the regularized Feynman rules assign to each diagram a integral that evaluates into some algebra $\mathcal{A}$. This is a linear map. If $\mathcal{A}$ is a k-algebra, the regularized Feynman rules define an algebra homomorphisms from $\mathcal{H}$ to $\mathcal{A}$.

Definition 3. The elements of $G(\mathcal{A})$, with $G = \text{Spec } \mathcal{H}$ are the generalized regularized Feynman rules for a QFT.

Regularized Feynman rules can be written as elements of $G(\mathcal{A})$ for some appropriately defined $\mathcal{A}$. These are the physical regularization theories. The general elements of the renormalization group $\phi \in G(\mathcal{A})$ need not have any physical interpretation at all.

Definition 4. The renormalization mass scale of a physical theory is represented by $\mathbb{R}_+$. It is the energy scale at which a physical theory is evaluated. In this paper, I follow the convention of [6] and complexify the energy scale, and write it $e^s \in \mathbb{C}^\times$ for $s \in \mathbb{C}$.

The regularized Feynman integrals are functions of the renormalization mass scale.

Definition 5. The renormalization scale action describes the dependence of the generalized regularized theory on the renormalization mass scale.

For example, consider $\phi_{dr} \in G(\mathcal{A})$, the dimensionally regularized Feynman rules for an (integer) d-dimensional scalar QFT. Let z be a complex parameter. For a given diagram Γ with $I(\Gamma)$ internal edges and $L(\Gamma)$ loops,

$$\phi_{dr}(z)(\Gamma) = A(d+z)^l \int_0^\infty \prod_{k=1}^{I(\Gamma)} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{L(\Gamma)} p_i^{d+z-1} dp_i \ .$$

Momentum cutoff regularization in the same theory gives

$$\varphi_{mc}(z)(\Gamma) = \int_{-\frac{1}{z}}^{\frac{1}{z}} \prod_{k=1}^{I(\Gamma)} \frac{1}{f_k(p_i, e_j)^2 + m^2} \prod_{i=1}^{L(\Gamma)} d^d p_i \ .$$

The action of the renormalization scale maps the momenta $p_i \to e^s p_i$ and thus

$$\phi_{dr}(z) \mapsto e^{sYz}\phi_{dr}(z)$$
$$\phi_{mc}(z) \mapsto \phi_{dr}(e^s z) \ .$$

Dimensionally regularized Feynman rules are elements of the group $\phi_{dr}(z) \in G(\mathbb{C}[z^{-1}][[z]])$. Momentum cutoff Feynman rules are in $\phi_{mc} \in G(\mathbb{C}[z^{-1}, \log(z)][[z]])$. For more details on this renormalization scale action, see [7, 3].

Definition 6. The action of the renormalization scale on a physical regularized QFT, $\phi \in G(\mathcal{A})$ defines a one parameter path in $G(\mathcal{A})$. This is called the renormalization flow of ϕ.

The action of the renormalization scale on a particular physical ϕ can be extended to an action of the renormalization scale on $G(\mathcal{A})$.

Definition 7. Let σ be an action of $\mathbb{C}$ on $G(\mathcal{A})$

$$\begin{aligned} \sigma : \mathbb{C} \times G(\mathcal{A}) &\to G(\mathcal{A}) \\ (s, \phi) &\to \sigma(s)(\phi) \ . \end{aligned}$$

In the examples above, I extend the dependence of dimensional regularization and momentum cutoff regularization to an generalized regularized theories as $\sigma_{dr}(s)(\phi) = e^{sYz}\phi(z)$ and $\sigma_{mc}(s)(\phi) = \phi(e^s z)$.

For physical reasons, one expects the paths defined by the renormalization scale to be integral; they are related to the solutions of the renormalization group equations, which describe the dependence of the observables of the theory on the energy scale. To mimick this mathematically, I am interested in extensions of the renormalization flows of physical theories to an action on $G(\mathcal{A})$ such that for each $\phi \in G(\mathcal{A})$, the renormalization flow, $\sigma(s)\phi$ is an integral path in $G(\mathcal{A})$. In other words the renormalization group action on $G(\mathcal{A})$ defines a one parameter family of diffeomorphisms on $G(\mathcal{A})$.

Definition 8. An action σ on $G(\mathcal{A})$ defines a renormalization group flow if it generates a one parameter family of diffeomorphisms on $G(\mathcal{A})$.

For the next theorem, let $\mathcal{A} = \mathbb{C}[z^{-1}, \log(z)][[z]]$. Both ϕ_{dr} and ϕ_{mc} can be written as elements of $G(\mathcal{A})$.

Proposition 2.1. *The actions σ_{dr} and σ_{mc} both define one parameter families of diffeomorphism on $G(\mathcal{A})$.*

Proof. Let $* \in \{mc, dr\}$. Since σ_* is an action on $G(\mathcal{A})$,

$$\sigma_*(s) \circ \sigma_*(u)(\phi(z)) = \sigma_*(s+u)\phi(z) \ .$$

The action σ_{dr} induces an automorphism on $G(\mathcal{A})$ [8]

$$e^{sYz}(\phi \star \psi) = e^{sYz}\phi \star e^{sYz}\psi \ .$$

Since the action is smooth, the result follows.

It is easy to check that σ_{mc} is a smooth map. It remains to check that it is bijective. To see surjectivity, notice that for any fixed $s \in \mathbb{C}$ and any $\phi(z) \in G(\mathcal{A})$, one can define $\phi'_s(z) = \phi(e^{-s}z)$, and

$$\phi(z) = \sigma_{mc}(s)\phi'_s(z) \ .$$

For injectivity, if there exists and $s \in \mathbb{C}$, and $\phi, \psi \in G(\mathcal{A})$, such that $\phi(sz)(\Gamma) = \psi(sz)(\Gamma)$ for every $\Gamma \in \mathcal{H}$, then $\phi(z)(\Gamma) = \psi(z)(\Gamma)$ for every $\Gamma \in \mathcal{H}$. This implies that $\phi(z) = \psi(z)$. $\square$

Definition 9. The physical β function for a renormalized QFT calculates the dependence of the coupling constant on the renormalization scale

$$\beta(g) = \frac{1}{\mu}\frac{dg}{d\mu} \ .$$

The physical β function is calculated perturbatively by loop number. In this Hopf algebraic picture of renormalization, a related object exists if the action σ defines a renormalization group flow on $G(\mathcal{A})$.

Theorem 2.2. *If σ defines a renormalization group flow on $G(\mathcal{A})$, it defines a complete a vector field $X_\sigma \in \mathfrak{X}(G(\mathcal{A}))$.*

Proof. By hypothesis, σ defines a one parameter family of diffeomorphisms on $G(\mathcal{A})$. Then $\sigma(s)\phi$ is an integral curve in $G(\mathcal{A})$ defined for all $s \in \mathbb{C}$, with $\sigma(0)\phi = \phi$. Define a the vector field

$$X_\sigma(\sigma(s)\phi) = \frac{d}{ds}\sigma(s)\phi \ .$$

This is is complete. □

Definition 10. The geometric β function for a renormalization group flow, σ, is defined

$$\beta_\sigma : G(\mathcal{A}) \to \mathfrak{g}(\mathcal{A})$$
$$\phi \to \phi^{-1} \star \frac{d}{ds}(\sigma(s)\phi)|_{s=0} =: \phi^{-1} \star X_\sigma(\phi) \ .$$

To see that $\beta_\sigma(\phi) \in \mathfrak{g}(\mathcal{A})$ for all $\phi \in G(\mathcal{A})$, note that $\beta_\sigma(\phi)$ is formed by left translating the vector $X_\sigma(\phi) \in T_\phi G(\mathcal{A})$ to $T_e G(\mathcal{A}) = \mathfrak{g}$.

Remark 1. In [4], the authors show that $z\beta_{\sigma_{dr}}(\phi) \in \mathfrak{g}(\mathbb{C})$, and is the generator of the one parameter subgroup of $G(\mathcal{A})$ defined $F_s(\phi) = \lim_{z\to 0} \phi^{-1} \star \sigma_{dr}(s)\phi$. This is a happy accidental property of dimensional regularization. It does not generalize to all regularization schemes or regularization group actions.

For more details on the geometric β function, especially in the case of σ_{mc} and σ_{dr}, see [3]. In the next section, we related the geometric β function to the Dynkin operator that appears in the study of dynamical systems.

3. Generalized Dynkin operators and Geometric β functions

Let $\mathcal{S}$ be a set and k a field of characteristic 0. Let $V = k[\{\mathcal{S}\}]$ be the vector space generated by this set. One can write $(V, [,])$ as a Lie algebra generated by S. The $T(V)$, the tensor algebra on V, is the universal enveloping algebra of V, $T(V) = \mathcal{U}(V)$. The classical (left) Dynkin operator D is a map

$$D : T(V) \to (V, [,])$$
$$x_1 \otimes \cdots \otimes x_n \to [x_1, [\cdots, [x_{n-1}, x_n] \cdots] \ .$$

Since $T(V) \simeq \mathcal{U}(V)$, $T(V)$ is isomorphic to a graded cocommutative Hopf algebra. Let Y be the grading operator. The elements of $\mathcal{S}$ are primitive, which defines comultiplication. Multiplication is defined by concatenation. The antipode is defined

$$S(x_1 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes \cdots \otimes x_1 \ .$$

Under this change of notation, the Dynkin operator $D = S \star Y$ [13]

$$S \star Y : \mathcal{U}(V) \to (V, [,]) \ .$$

The grading operator Y is a derivation on $\mathcal{U}(V)$. Let $G = \exp(V)$. The Baker-Campbell-Hausdorff (BCH) formula provides an inverse map from $G \to V$. The Dynkin operator defines a closed form for the BCH formula [9]

$$\log(\exp X \exp Y) = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\le i\le n}} \frac{(\sum_{i=1}^n (r_i + s_i))^{-1}}{r_1! s_1! \cdots r_n! s_n!} D(X^{r_1}Y^{s_1}X^{r_2}Y^{s_2} \ldots X^{r_n}Y^{s_n}) \ .$$

In fact, the Dynkin operator, D, defines a bijection from G to V. I call this the Dynkin map.

In [9], the authors show that given any derivation δ on a graded commutative Hopf algebra, $\mathcal{H}$ the map $D_\delta = S \star \delta$ defines a bijection between $G(\mathcal{A}) = \mathrm{Hom}_{k\ alg}(\mathcal{H}, \mathcal{A})$ and $\mathfrak{g}(\mathcal{A}) = \mathrm{Lie}(G(\mathcal{A}))$, by defining

$$D_\delta(\phi)(\mathcal{H}) := \phi(S \star \delta)(\Delta(h)) \ .$$

It is easy to check that the grading operator Y is a derivation on $\mathcal{H}$. Using the notation established in this paper, they show that $z\beta_{\sigma_{dr}}$ corresponds to right composition by the Dynkin map $D_Y = S \star Y$,

$$z\beta_{\sigma_{dr}}(\phi) = \phi^{-1} \star Y\phi = \phi \circ D_Y \ .$$

I generalize this finding.

Theorem 3.1. *The geometric β function, β_σ is a generalized Dynkin map, D_{X_σ} from $G(\mathcal{A})$ to $\mathfrak{g}(\mathcal{A})$*

$$\beta_\sigma(\phi) = \phi^{-1} \star X_\sigma(\phi) = \phi \circ D_{X_\sigma} \ .$$

Proof. The map β_σ is defined by the vector field X_σ on $G(\mathcal{A})$. Vector fields on a Lie group define derivations on the algebra of regular functions on that group. Since $G = \mathrm{Spec}\, \mathcal{H}$, the algebra of regular function $k[G] \simeq \mathcal{H}$. Therefore X_σ defines a derivation on $\mathcal{H}$, call it δ_σ. Specifically, for $h \in k[G]$,

$$X_\sigma(\phi) \leftrightarrow \delta_\sigma(h)(\phi) := \frac{d}{ds} h(\sigma_s(\phi))|_{s=0} \ .$$

Recall that the product on $k[G]$ is defined pointwise

$$hh'(\phi) = h(\phi)h'(\phi) \ .$$

It is easy to check that δ_σ is a derivation

$$\delta_\sigma(hh')(\phi) = \frac{d}{ds}\left(hh'(\sigma_s(\phi))\right)|_{s=0} = \frac{d}{ds}\left(h(\sigma_s(\phi))h'(\sigma_s(\phi))\right) =$$
$$\frac{d}{ds}\left(h(\sigma_s(\phi))\right)|_{s=0} h'(\phi) + h(\phi)\frac{d}{ds}\left(h(\sigma_s(\phi))\right)|_{s=0} = (\delta_\sigma(h)h')(\phi) + (h\delta_\sigma(h'))(\phi) \ .$$

The first equality is from the definition of δ_σ, the second from the definition of $k[G]$. Under this set of definitions,

$$\beta_\sigma(\phi)(h) = \phi^{-1} \star X_\sigma(\phi)(h) = \phi \circ (S \star \delta_\sigma)(\Delta h) \ .$$

In other words, $\beta_\sigma = \phi \circ D_{X_\sigma}$. ☐

Remark 2. Note that this implies that the geometric β function β_σ defines a set bijection from $G(\mathcal{A})$ to $\mathfrak{g}(\mathcal{A})$.

Corrollary 3.2. *The geometric β function*

$$\beta_\sigma : G(\mathcal{A}) \to \mathfrak{g}(\mathcal{A}) \ ,$$

is defined by the Maurer-Cartan connection on the Lie group $G(\mathcal{A})$ contracted with X_σ.

Proof. The Maurer-Cartan connection is a $\mathfrak{g}(\mathcal{A})$ valued one form defined

$$\theta : \phi^{-1} \star d\phi$$

for $\phi \in G(\mathcal{A})$. Contracting with a vector field, X_σ

$$\langle X_\sigma(\phi), \theta \rangle = \phi^{-1} \star X_\sigma(\phi) = \beta_\sigma(\phi) \ .$$

☐

In [11], the authors define a generalization of the classical Dynkin operator, $D_\delta = S \star \delta$ that is defined by a Lie derivation δ on a free Lie algebra $\mathfrak{g}$

$$D_\delta : \mathcal{U}(\mathfrak{g}) \to \mathfrak{g} \ .$$

It is a Lie idempotent in the sense that if $x \in \mathfrak{g}$, then $D_\delta(x) = \delta(x)$.

In the context of renormalization, the Hopf algebra of Feynman diagrams $\mathcal{H}$, is of finite type. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is freely generated, and the graded dual, $\mathcal{H}^\vee \simeq \mathcal{U}(\mathfrak{g})$. Vector fields on G exactly define Lie derivatives on $\mathfrak{g}$. This gives the following theorem.

Theorem 3.3. *The renormalization group flow defining action σ defines a generalized Dynkin operator in the sense of* [11].

Proof. If the action σ defines a renormalization group flow on $G(\mathcal{A})$, then it defines a one parameter family of diffeomorphisms on $G(\mathcal{A})$, and thus a complete vector field $X_\sigma \in \mathfrak{X}(G(\mathcal{A}))$. The derivative δ_σ on $\mathcal{H}$ is exactly the Lie derivative on $\mathfrak{g}(\mathcal{A})$ defined by X_σ.

The action σ induces a path through $\mathfrak{g}(\mathcal{A})$ defined by the map β_σ. Since β_σ is a bijection from $G(\mathcal{A})$ from $\mathfrak{g}(\mathcal{A})$, for any $\alpha \in \mathfrak{g}(\mathcal{A})$, one can find a $\phi \in G(\mathcal{A})$ such that $\alpha = \beta_\sigma(\phi)$. The action of σ on $G(\mathcal{A})$ lifts to an action on $\mathfrak{g}(\mathcal{A})$ as

$$\sigma(s)(\alpha) = \sigma(s)(\phi)\frac{d}{ds}(\sigma(s)(\phi)) \ .$$

The Lie derivative δ_σ gives

$$\delta_\sigma(\alpha)(h) = \alpha(\delta_\sigma(h)) = \frac{d}{ds}\sigma(s)(\alpha)(h) \ .$$

Let $\gamma \in \mathfrak{g}(\mathcal{A})$. Writing $\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$,

$$\delta_\sigma(\alpha \star \gamma)(h) = \frac{d}{ds}(\sum_{(h)} \sigma(s)(\alpha)(h_{(1)})\sigma(s)(\gamma)(h_{(2)}))$$

$$= \frac{d}{ds}(\sigma(s)(\alpha)) \star \sigma(s)(\gamma)(h) + \sigma(s)(\alpha)\frac{d}{ds}(\sigma(s)(\gamma))(h) = \delta_\sigma(\alpha) \star \gamma(h) + \alpha\delta_\sigma(\gamma)(h) \ .$$

□

To summarize, I relate the generalizations of the Dynkin map defined in [9] and the generalized Dynkin map defined in [11].

Theorem 3.4. *Each action σ on $G(\mathcal{A})$ that defines a one parameter family of diffeomorphism on $G(\mathcal{A})$ and thus induces the vector field X_σ, defines a generalized Dynkin operator*

$$D_{X_\sigma} : \mathcal{U}(\mathfrak{g}) \to \mathfrak{g} \ .$$

The associated geometric β function β_σ defines a generalized Dynkin map defined by the Maurer-Cartan connection, θ,

$$\begin{aligned} \beta_\sigma : G(\mathcal{A}) &\to \mathfrak{g}(\mathcal{A}) \\ \phi &\to < X_\sigma(\phi), \theta > \ . \end{aligned}$$

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References

1. Susama Agarwala, *The β-function over curved space-time under ζ-function regularization*, (2009), arXiv:0909.4122.
2. ______, *A perspective on renormalization*, Letters in Mathematical Physics **93** (2010), 187–201, arXiv:0909.4117.
3. ______, *Geometrically relating momentum cut-off and dimensional regularization*, International Journal of Geometric Methods in Mathematical Physics **10** (2013), arXiv:1107.5533.
4. Alain Connes and Dirk Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem I: The Hopf algebra structure of graphs and the main theorem*, Communications in Mathematical Physics **210** (2001), 249–273, arXiv:hep-th/9912092v1.
5. ______, *Renormalization in Quantum Field Theory and the Riemann-Hilbert problem II: The β function, diffeomorphisms and renormalization group*, Communications in Mathematical Physics **216** (2001), 215–241, arXiv:hep-th/0003188v1.
6. Alain Connes and Matilde Marcolli, *Quantum fields and motives*, Journal of Geometry and Physics **56** (2006), 55–85.
7. ______, *Noncommutative geometry, quantum fields, and motives*, American Mathematical Society, USA, 2008.
8. Kurusch Ebrahimi-Fard and Dominique Manchon, *On matrix differential equations in the Hopf algebra of renormalization*, Advances in Theoretical and Mathematical Physics **10** (2006), 879–913, arXiv:math-ph/0606039v2.
9. Kurush Ebrahimi-Fard, Jose M. Garcia-Bondiá, and Frédéric Patras, *A Lie theoretic approach to renormalizaton*, Communications in Mathematical Physics **276** (2007), 519–549, arXiv:hep-th/0609035.

10. Dirk Kreimer, *Anatomy of a gauge theory*, Annals of Physics **321** (2006), 2757 – 2781, arXiv:hep-th/0509135.
11. Frederic Menous and Frédéric Patras, *Logarithmic derivatives and generalized dynkin operators*, arXiv:1206.4990, 2012.
12. Walter D. van Suijlekom, *Renormalization of gauge fields: A Hopf algebra approach*, Communications in Mathematical Physics **276** (2007), 773–798, arXiv:hep-th/0610137v1.
13. Wilhelm von Waldenfels, *Zur charakterisierung liescher elemente in freien algebren*, Archiv der Mathematik **17** (1966), 44–48.

Hamburg University
E-mail address: susama.agarwala@math.uni-hamburg.de
URL: http://www.its.caltech.edu/∼susama/